Individual Competition
27th August, 2015

English version

Problem I–1

Find all surjective functions $f\colon \mathbb{N} \to \mathbb{N}$ such that for all positive integers a and b, exactly one of the following equations is true:

$$f(a) = f(b),$$
$$f(a+b) = \min\{f(a), f(b)\}.$$

Remarks: $\mathbb{N}$ denotes the set of all positive integers. A function $f\colon X \to Y$ is said to be surjective if for every $y \in Y$ there exists $x \in X$ such that $f(x) = y$.

Problem I–2

Let $n \geqslant 3$ be an integer. An *inner diagonal* of a *simple n-gon* is a diagonal that is contained in the n-gon. Denote by $D(P)$ the number of all inner diagonals of a simple n-gon P and by $D(n)$ the least possible value of $D(Q)$, where Q is a simple n-gon. Prove that no two inner diagonals of P intersect (except possibly at a common endpoint) if and only if $D(P) = D(n)$.

Remark: A simple n-gon is a non-self-intersecting polygon with n vertices. A polygon is not necessarily convex.

Problem I–3

Let $ABCD$ be a cyclic quadrilateral. Let E be the intersection of lines parallel to AC and BD passing through points B and A, respectively. The lines EC and ED intersect the circumcircle of AEB again at F and G, respectively. Prove that points C, D, F, and G lie on a circle.

Problem I–4

Find all pairs of positive integers (m, n) for which there exist relatively prime integers a and b greater than 1 such that

$$\frac{a^m + b^m}{a^n + b^n}$$

is an integer.

Time: 5 hours
Time for questions: 60 min
Each problem is worth 8 points.
The order of the problems does not depend on their difficulty.

Problem T–1

Prove that for all positive real numbers a, b, c such that $abc = 1$ the following inequality holds:

$$\frac{a}{2b+c^2} + \frac{b}{2c+a^2} + \frac{c}{2a+b^2} \leqslant \frac{a^2+b^2+c^2}{3}.$$

Problem T–2

Determine all functions $f\colon \mathbb{R}\setminus\{0\} \to \mathbb{R}\setminus\{0\}$ such that

$$f(x^2yf(x)) + f(1) = x^2f(x) + f(y)$$

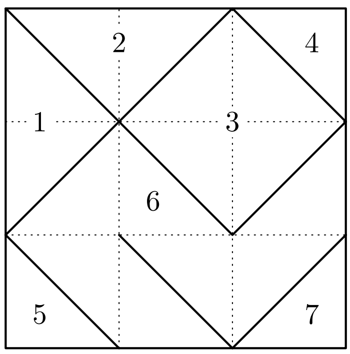
holds for all nonzero real numbers x and y.

Problem T–3

There are n students standing in line in positions 1 to n. While the teacher looks away, some students change their positions. When the teacher looks back, they are standing in line again. If a student who was initially in position i is now in position j, we say the student moved for $|i-j|$ steps. Determine the maximal sum of steps of all students that they can achieve.

Problem T–4

Let N be a positive integer. In each of the N^2 unit squares of an $N \times N$ board, one of the two diagonals is drawn. The drawn diagonals divide the $N \times N$ board into K regions. For each N, determine the smallest and the largest possible values of K.



Example with $N = 3$, $K = 7$

Time: 5 hours
Time for questions: 60 min
Each problem is worth 8 points.
The order of the problems does not depend on their difficulty.

Problem T–5

Let ABC be an acute triangle with $AB > AC$. Prove that there exists a point D with the following property: whenever two distinct points X and Y lie in the interior of ABC such that the points B, C, X, and Y lie on a circle and

$$\angle AXB - \angle ACB = \angle CYA - \angle CBA$$

holds, the line XY passes through D.

Problem T–6

Let I be the incentre of triangle ABC with $AB > AC$ and let the line AI intersect the side BC at D. Suppose that point P lies on the segment BC and satisfies $PI = PD$. Further, let J be the point obtained by reflecting I over the perpendicular bisector of BC, and let Q be the other intersection of the circumcircles of the triangles ABC and APD. Prove that $\angle BAQ = \angle CAJ$.

Problem T–7

Find all pairs of positive integers (a, b) such that

$$a! + b! = a^b + b^a.$$

Problem T–8

Let $n \geqslant 2$ be an integer. Determine the number of positive integers m such that $m \leqslant n$ and $m^2 + 1$ is divisible by n.